

A Fermion Crystal with Quantum Coherence

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When two-dimensional electrons are subjected to a very strong magnetic field, they are believed to form a triangular Wigner crystal. We demonstrate that, in the entire crystal phase, this crystal is very well represented by a composite-fermion-crystal wave function, revealing that it is not a simple Hartree-Fock crystal of electrons but an inherently quantum mechanical crystal characterized by a non-perturbative binding of quantized vortices to electrons, which establishes a long range quantum coherence in it. It is suggested that this has qualitative consequences for experiment.

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Wigner argued that electrons form a crystal when the interaction energy of electrons dominates over their kinetic energy [1]. One such situation occurs when a two-dimensional electron system is exposed to a strong magnetic field (B). The kinetic energy is quantized into Landau levels under a magnetic field, and the number of occupied Landau levels is called the filling factor, $\nu = \rho h c / e B$, ρ being the two-dimensional electron density. At sufficiently strong magnetic fields, when all electrons fall into the lowest Landau level ($\nu < 1$), the kinetic energy is no longer relevant, and the nature of the state is determined solely by the Coulomb interaction. One might expect a crystal for any $\nu < 1$, but for a range of filling factors the system condenses into a quantum Hall liquid, characterized by dissipationless transport and precisely quantized plateaus of Hall resistance [2]. There are strong indications, however, that a crystal occurs at sufficiently low ν , and its properties have been studied in many experimental [3, 4, 5, 6, 7, 8, 9, 10, 11] and theoretical [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] works.

Our understanding of the physics of the FQHE liquid is secure in part because the microscopic wave functions [24, 25] incorporating this physics have been shown to be extremely accurate through comparisons against *exact* wave functions known for systems containing a finite number of electrons. The liquid wave functions become progressively worse with decreasing ν , consistent with the expectation that a crystal becomes energetically favorable at some point. Indeed, a Hartree-Fock wave function describing an “electron crystal” (EC) provides a better approximation for the ground state below certain filling factor. However, the electron crystal is not particularly good quantitatively. The aim of this paper is to show that the low ν phase is strikingly accurately described as a quantum crystal of composite fermions, where composite fermions are bound states of electrons and quantized vortices. The composite-fermion (CF) crystal wave function offers a quantitative description of the crystal state at a similar level of accuracy as the CF liquid wave functions do for the FQHE liquid. One might have expected residual CF correlations to sur-

vive in the crystal phase in the proximity of the phase boundary separating the CF liquid and the crystal, but we find that the CF crystal is realized even deep inside the crystal phase, down to arbitrarily low fillings. We estimate the temperature below which thermal fluctuations should not destroy the quantum effects due to the formation of composite fermions in the crystal.

The gaussianly localized wave packet for an electron in the lowest Landau level localized at $\mathbf{R} = (X, Y)$ is given by [13]

$$\phi_{\mathbf{R}}(\mathbf{r}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{4}(\mathbf{r} - \mathbf{R})^2 + \frac{i}{2}(xY - yX)\right) \quad (1)$$

where the magnetic length, $l_0 = \sqrt{\hbar c / e B}$, has been taken as the unit of length. The wave function for the electron crystal (EC) is constructed by placing electrons on a triangular lattice \mathbf{R}_j , the lowest energy solution for the classical problem, and then antisymmetrizing the product: [13]

$$\Psi^{EC} = \frac{1}{\sqrt{N!}} \sum_P \epsilon_P \prod_{j=1}^N \phi_{\mathbf{R}_j}(\mathbf{r}_{Pj}) \quad (2)$$

where the sum is over all permutations P and ϵ_P is +1 for even permutations and -1 for odd permutations. With the lattice constant $a = (4\pi/\sqrt{3}\nu)^{1/2}l_0$, the overlap integral between nearest neighbor electron wave functions [13], $\exp(-a^2/2l_0^2) = \exp(-3.627/\nu)$, decays rapidly with decreasing ν . We will work with the symmetric gauge, $\mathbf{A} = (B/2)(-y, x, 0)$, for which the total angular momentum L is a good quantum number. Because the wave function Ψ^{EC} is not an eigenstate of angular momentum, we follow the method of Yannouleas and Landman [26] to project it onto a definite L , denoting the resulting wave function Ψ_L^{EC} . Such projection amounts to creating a rotating crystal, implying that the crystalline structure is not apparent in the density but in the pair correlation function. The explicit expression for Ψ_L^{EC} is given in Ref. 26.

Following the standard procedure of the composite fermion (CF) theory [25, 27], we construct the following

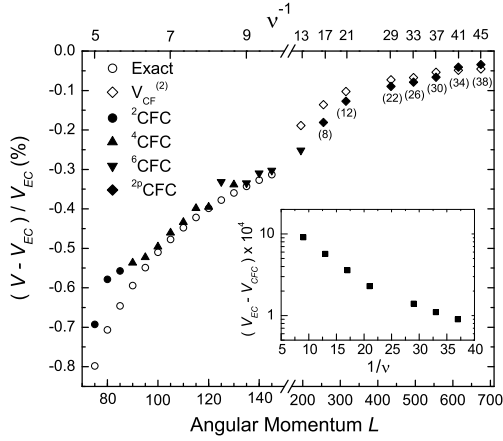


FIG. 1: The correlation energy of the optimal CF crystal, i.e., the % deviation of its Coulomb energy from the Coulomb energy of the uncorrelated electron crystal, for $N = 6$ particles. The superscript $2p$ on ^{2p}CFC indicates the vortex quantum number of composite fermions. The energy of the electron crystal for $L > 400$ is taken from Yannouleas and Landman [26]. The deviation of the exact energy from the electron crystal energy is also shown for $L \leq 145$; for larger angular momenta, where the exact energy is not available, we show an accurate approximation, $V_{CF}^{(2)}$ (explained in the text), as an independent reference. For $2p > 6$, the number of vortices carried by composite fermions is shown in brackets near the diamond. The energy difference per particle between the electron and the CF crystals is given in the inset, quoted in units of $e^2/\epsilon l_0$, where l_0 is the magnetic length and ϵ the dielectric constant of the host semiconductor.

wave function:

$$\Psi_L^{2p\text{CFC}} = \prod_{j < k} (z_j - z_k)^{2p} \Psi_{L^*}^{EC}, \quad (3)$$

$$L^* = L - pN(N-1). \quad (4)$$

It is interpreted as a CF crystal (CFC), because the Jastrow factor $\prod_{j < k} (z_j - z_k)^{2p}$ binds $2p$ quantized vortices to each electron in Ψ^{EC} to convert it into a composite fermion. The composite fermions of different flavors are denoted by ^{2p}CF , and their crystals by ^{2p}CFC . A variational wave function analogous to that in Eq. (3) was considered earlier by Yi and Fertig [20] and was shown to have lower energy than the Lam-Girvin [15] wave function in the filling factor range $0.1 < \nu < 0.2$. Narevich, Murthy, and Fertig [21] used a Hamiltonian formulation of composite fermions to estimate gaps and shear modulus on either side of the $\nu = 1/5$ quantum Hall state. However, it is not known how close $\Psi_L^{2p\text{CFC}}$ is to the actual ground state and what is the filling factor range of its applicability. It is noted that Jastrow-type wave functions were earlier used to estimate the effect of zero point motion in the ^4He crystal [28].

Comparisons with exact solutions have served as a rigorous and unbiased means for establishing the physics of various strongly correlated states in the lowest Landau level. We now subject $\Psi^{2p\text{CFC}}$ to such tests. Exact wave functions can be obtained (using the Lanczos method) for up to $N = 7$ particles in the low- ν region of interest. We will present below detailed results for $N = 6$; the study of $N = 5$ and $N = 7$ particles is consistent with our conclusions below. The filling factor of the finite system will be defined by the expression $\nu = N(N-1)/2L$, which gives the correct value of ν in the thermodynamic limit. For $N = 6$, the lowest energy classical configuration has one particle at the center, with the remaining five forming a ring around it [29]. The wave functions in Eq. (3) for $2p \neq 0$ have rather complicated correlations built into them, but the interaction energy per particle,

$$V = \frac{1}{N} \frac{\langle \Psi_L^{2p\text{CFC}} | \sum_{j < k} \frac{e^2}{\epsilon r_{jk}} | \Psi_L^{2p\text{CFC}} \rangle}{\langle \Psi_L^{2p\text{CFC}} | \Psi_L^{2p\text{CFC}} \rangle}, \quad (5)$$

can be evaluated by the Metropolis Monte Carlo method at least for many large values of $2p$ (the computation time increases rapidly as $2p$ is reduced). The total energy also has contributions from electron-background and background-background interactions, but these terms are the same for different crystal wave functions for a given L , so are not relevant for comparisons.

Fig. 1 shows the correlation energy of the optimal CF crystal, defined as the deviation of its energy from that of the uncorrelated electron crystal. (The computationally accessible range of $2p$ allows us to determine the minimum CFC energy. The only exceptions are the largest two values of L , where we show the energy at the smallest $2p$ studied; the minimum CF energy may be still lower here.) The exact correlation energy is also shown for $L \leq 145$. For $L > 145$, the dimension of the Fock space (D) is too large for an exact treatment. As an independent reference point, we obtain an accurate approximation to the exact energy by the method of “CF diagonalization,” wherein the Coulomb Hamiltonian is diagonalized in a correlated CF basis, the dimension of which is much smaller than the dimension of the full basis needed for the exact state; gradually increasing the basis size gives an increasingly better approximation [30]. Fig. 1 quotes $V_{CF}^{(2)}$ (using the notation in Ref. 30), obtained with a correlated CF basis of dimension 150. $V_{CF}^{(2)}$ has been shown to be very precise [30]: for the six particle system it is within 0.02% of the exact energy for $L \leq 145$ and we expect similar level of accuracy for higher L as well.

The minimum energy for all L is obtained at a non-zero value of $2p$, which establishes that the CF crystal provides a better variational state than the electron crystal. Most significantly, the CF crystal is essentially the exact solution for $\nu \leq 1/7$ ($L \geq 105$). For $100 < L < 145$, the energy of the optimal CF crystal is approximately

TABLE I: The last three columns give the overlaps of CF crystal (CFC), electron crystal (EC), and Laughlin's wave function with the exact ground state wave function at several filling factors ν . The overlap is defined as $|\langle \Psi^{trial} | \Psi^{exact} \rangle|^2 / \langle \Psi^{trial} | \Psi^{trial} \rangle \langle \Psi^{exact} | \Psi^{exact} \rangle$. The second column gives D , the dimension of the basis space for $N = 6$ electrons, and L is the total angular momentum of the state.

ν (L)	D	CFC	EC	Laughlin
1/5 (75)	19858	0.891	0.645	0.701
1/7 (105)	117788	0.994	0.723	0.504
1/9 (135)	436140	0.988	0.740	0.442

within 0.02% of the exact energy. Tables I and II show how Laughlin's liquid wave function, Ψ^{EC} , and Ψ^{CFC} compare with the exact wave function for $\nu = 1/5$, $1/7$ and $1/9$. As indicated earlier, the liquid wave function worsens and Ψ^{EC} improves with decreasing ν , but neither is very good at small fillings. In contrast, Ψ^{CFC} is surprisingly close to the exact state. Its overlap with the exact wave function is $\sim 99\%$ for $\nu = 1/7$ and $1/9$, while its energy deviates from the exact energy by 0.016% and 0.006%, respectively. To appreciate the significance of these comparisons, we note: (i) The exact state is a linear combination of a large number of Slater determinant basis functions (see Table I), involving $D - 1$ parameters, and yet, a single CFC wave function captures its physics almost exactly. (ii) The CFC wave function for $\nu \leq 1/7$ is more accurate than accepted wave functions for the FQHE liquid states. For example, for the system of six particles at $\nu = 1/3$ in the disk geometry, the energy of Laughlin's wave function [24] is off by 0.15% and its overlap with the exact state is 0.964, in spite of the fact that the dimension of the Fock space is much smaller ($D = 1206$). For larger L , the energy of the CFC is lower than $V_{CF}^{(2)}$, with the possible exception of the last two points, where we may not have the optimal CFC. We take these comparisons as a definitive verification of the formation of a CF crystal at low ν .

Because every particle sees quantized vortices on every other particle, the formation of composite fermions implies a long range quantum coherence in the crystal phase. To get a feel for how the binding of vortices to electrons affects the inter-particle correlations, we show in Fig. 2 the pair correlation function $g(x)$ for several candidate wave functions as well as the exact ground state for $\nu = 1/7$; $g(x)$ is the probability of finding a pair of particles at an arc distance x on a circle of radius R . (R is chosen to match the distance of a particle in the parent classical crystal from the center of the disk.) The result shows that the crystalline correlations are slightly weakened by the formation of composite fermions. It is perhaps counter-intuitive that such an effect should lead to a lower energy even at very low fillings.

TABLE II: Interaction energies per particle for the exact ground state, the CF crystal (CFC), the electron crystal (EC), and Laughlin's wave function for six particles at several filling factors. The uncertainty in the last digit from Monte Carlo sampling is given in parentheses.

ν (L)	exact	CFC	EC	Laughlin
1/5 (75)	2.2019	2.2042(5)	2.2196	2.2093(2)
1/7 (105)	1.8533	1.8536(2)	1.8622	1.8617(2)
1/9 (135)	1.6305	1.6306(1)	1.6361	1.6388(1)

Of interest is the nature of the thermodynamic state, obtained in the limit $N \rightarrow \infty$ at a fixed filling factor. Finite size studies do not necessarily provide a reliable insight into the thermodynamic state. For example, at $\nu = 1/5$ the CFC gives the best description for $N = 6$ but the thermodynamic state here is known to be a liquid [3, 4]. However, when an extremely precise and unambiguous description of the finite N state is obtained, as is the case at $\nu \leq 1/7$, we consider that to be a strong indication for the nature of the state in the thermodynamic limit. In any case, even though our finite N study cannot give the precise ν value where a transition from liquid to crystal takes place, it does make a compelling case that whenever the thermodynamic state is a crystal, it is a crystal of composite fermions, even in regions of the phase diagram far from the CF liquid.

The quantum character of the crystal is not fragile, and ought to be observable at presently attainable temperatures, even at very small ν . The energy difference per particle, $V^{CFC} - V^{EC}$, shown in the inset of Fig. 1, gives a crude estimate for the temperature below which the quantum nature of the crystal should be robust to thermal fluctuations. The relevant temperatures appear to be well within the present experimental reach – for example, for parameters appropriate for GaAs, the quantum crystal regime is estimated to be below ≈ 25 mK (at $B = 25$ T) even at $\nu = 1/33$. From the N dependence, we have estimated that the energy difference shown in the inset underestimates the thermodynamic energy difference by approximately a factor of two. It is interesting to note that even as the energy difference between the CF and the electron crystals decreases as $\nu \rightarrow 0$, $2p$ continues to rise. Thus, CF flavors of up to very high $2p$ are predicted to occur in the crystal state. In the liquid phase, ^2CFs and ^4CFs have definitely been observed, and there is evidence also for ^6CFs and ^8CFs at relatively high temperatures. [10]

Quantum crystals of neutral bosons have been studied previously, where quantum mechanical exchange due to a significant overlap between the neighboring atoms is responsible for the quantum behavior, including a possible “supersolid” phase in which Bose-Einstein condensation can coexist with crystallinity [31, 32]. In the lowest Lan-

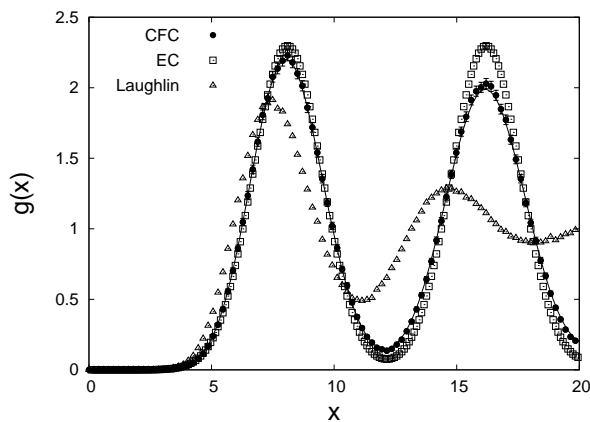


FIG. 2: The pair correlation functions for the CF crystal (solid circles), the electron crystal (empty squares), and Laughlin's wave function (empty triangles) on a circle of radius $R = 6.445l_0$ for six particles at $\nu = 1/7$. The solid line shows the exact pair correlation function.

dau level, the overlap between (uncorrelated) electron wave packets at neighboring sites is negligible in the filling factor region of interest (the overlap integral is 10^{-15} for $\nu = 1/9$). The quantum nature of the CF crystal owes its origin to the long range Coulomb interaction.

Given that the CF liquid behaves qualitatively differently from an electron liquid, one may ask in what ways the properties of the CF crystal are distinct from those of an electron crystal. We mention here a few examples where the CFC can provide natural explanations for certain experimental facts, although further work will be needed to make the connection with experiment more direct and to clarify other possible implications. The issue is obviously relevant to experiments that exhibit transitions between the liquid and crystal phases. Re-entrant transitions between the FQHE liquid and an insulating state, thought to be a pinned crystal, have been observed [3, 4] in going from $\nu = 2/5$ to $\nu = 1/5$. In the filling factor range $1/5 > \nu > 1/9$, the low temperature insulating state melts into a CF liquid upon raising temperature [10], as indicated by the appearance of FQHE like structure. These observations become less baffling knowing that the crystal is itself made of composite fermions rather than electrons, thus requiring a less drastic reorganization of the state at the transition. Another result, perhaps puzzling for an electron crystal, is that the Hall resistance of the pinned crystal is close to the value it would have for a liquid [33]. If the current is carried by composite fermions instead, then the Hall voltage induced by the accompanying vortex current (the vortices effectively behave as magnetic flux quanta [25, 34]), through an effective Faraday effect, is roughly consistent with the observation. (Zheng and Fertig [19] considered a similar mechanism for transport by correlated interstitial defects.) The unexpectedly small

activation energy in the crystalline state, compared to theoretical predictions based on an electron crystal, as well as its non-monotonic filling factor dependence has also been rationalized in terms of a CF crystal [20, 21].

In summary, we have shown that the two-dimensional electron system in high magnetic fields provides an example of an intrinsically quantum mechanical crystal of fermions with non-trivial correlations. Partial support of this research by the National Science Foundation under grant no. DMR-0240458 is gratefully acknowledged.

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